

# Investigation 5

## ACE Assignment Choices



### Problem 5.1

Core 1–7, 54

Other Applications 8, 9; Connections 44, 45; Extensions 52, 53, 55; unassigned choices from previous problems

### Problem 5.2

Core 10–27, 31

Other Applications 28–30, 32–41; Connections 46–48; Extensions 56–63; unassigned choices from previous problems

### Problem 5.3

Core 42, 43

Other Connections 49–51; unassigned choices from previous problems

**Adapted** For suggestions about adapting Exercise 31 and other ACE exercises, see the *CMP Special Needs Handbook*.

**Connecting to Prior Units** 51: *Bits and Pieces I*; 46, 47: *Filling and Wrapping* and *Stretching and Shrinking*; 48, 49: *Prime Time*

## Applications

9. The ones digits for powers of 7 cycle through 7, 9, 3, and 1. Because the exponent, 100, is a multiple of 4, the ones digit will match the fourth number in the cycle, which is 1.
6. The only possibility for the ones digit for a power of 6 is 6.
9. The same reasoning from ACE Exercise 1 holds because the ones digit of 17 is 7.  $7^{100}$  and  $17^{100}$  have the same ones digit.
1. To get successive powers of 31, you repeatedly multiply by 31, and the ones digit is always 1 times the previous ones digit. So the ones digit is always a power of 1, or 1.
6. The possibilities for the ones digit when the base is 12 are the same as when the base is 2. So, the ones digits cycle through 2, 4, 8, and 6.

Because the exponent, 100, is a multiple of 4, the ones digit will be the fourth number in the cycle, which is 6.

7. The possibilities for  $a$  include 3 and 7 because the ones digit in 823,543 is 3. Because 823,543 has 6 digits and the power is 7, 3 is too small, so  $a$  must equal 7.
11.  $a$  could be any number with a ones digit of 1, 3, 7, or 9. Because 1,771,561 has 7 digits and  $10^6 = 1,000,000$  has 7 digits,  $a$  must be greater than 10 but close to 10, so  $a$  is 11.
- Possible answer: The ones digit is 9. You can find  $3^{15}$ , which is 14,348,907. The expression  $3^{30}$  is equivalent to a product string of thirty 3s, which is the same as the product of two strings of fifteen 3s. So,  $3^{30} = 3^{15} \times 3^{15}$ . Therefore, the ones digit of  $3^{30}$  is the same as the ones digit of  $7 \times 7$  (the ones digit of  $3^{15}$  times itself).
- C. Square numbers have a ones digit of 1, 4, 9, 6, 5, or 0, so 1,392 is not a square number. However, 289 and 10,000 could be square numbers because they end in 0 and 9. In fact,  $17^2 = 289$  and  $100^2 = 10,000$ .

- |               |              |
|---------------|--------------|
| 10. 10 zeros  | 11. 50 zeros |
| 12. 100 zeros | 13. 6        |
| 14. 7         | 15. $7^{10}$ |
| 16. $8^{10}$  | 17. $6^9$    |

**Note to the Teacher** Students may use their calculators for Exercises 15–17, but they should be able to use the rules of exponents and some estimation or mental arithmetic. The reasoning for  $6^9 > 9^6$ , for example, might look like this:

$$6^9 = (2 \times 3)^9 = 2^9 \times 3^9 = 2^9 \times 3^3 \times 3^6$$

and

$$9^6 = (3 \times 3)^6 = 3^6 \times 3^6 = 3^3 \times 3^3 \times 3^6$$

Comparing these comes down to comparing  $2^9$  and  $3^3$ . Because  $2^9 > 3^3$ ,  $6^9 > 9^6$ .

18. G
19.  $40^6$
20.  $7^{15}$
21.  $8^5$
22. True; this is an example of  $a^m \times a^n = a^{m+n}$
23. False;  $2^3 \cdot 3^2 = 8 \cdot 9 = 72$  and  $72 \neq 6^5$
24. True;  
 $3^8 = (3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3) = (3^2)^4 = 9^4$
25. False;  $4^3 + 5^3 = 64 + 125 = 189$  and  $189 \neq 9^3$
26. True; by the distributive property,  
 $2^3(1 + 2^2) = (2^3 \cdot 1) + (2^3 \cdot 2^2) = 2^3 + 2^5$ .  
 Or, students may evaluate both sides and find that both sides are equal to 40.
27. False.  $\frac{5^{12}}{5^4} = 5^8 \neq 5^3$
28. C
29. 8. Because  $4^{15} \times 3^{15} = (4 \times 3)^{15} = 12^{15}$ , the ones digit is the same as the ones digit for  $2^{15} = 32,768$ .
30. 8. The ones digits for powers of 7 occur in cycles of 7, 9, 3, 1. Because 15 divided by 4 leaves a remainder of 3, the ones digit of  $7^{15}$  is the third digit in the cycle, which is 3. The ones digits for powers of 4 occur in cycles of 4, 6. Because 20 is evenly divisible by 2, the ones digit of  $4^{20}$  is the second digit in the cycle, which is 6. So, the ones digit of  $7^{15} \times 4^{20}$  is the ones digit of  $3 \times 6 = 18$ .
31. a. Manuela is correct because  $2^{10} = 1,024$  and  $2^4 \times 2^6 = 16 \times 64 = 1,024$ .  
 b. Possible answers:  
 $2^2 \times 2^8 = 4 \times 256 = 1,024$   
 $2^3 \times 2^7 = 8 \times 128 = 1,024$   
 $2^2 \times 2^2 \times 2^6 = 4 \times 4 \times 64 = 1,024$   
 c. 4,096. Because  $2^7 = 128$  and  $2^5 = 32$ ,  $2^{12}$  is  $2^7 \times 2^5 = 128 \times 32 = 4,096$ .  
 d. It works for other cases because you are just using the associative property of multiplication. She is grouping strings of the same factor into two groups.
32. Yes; it has exactly 10 factors of 1.25.
33. Yes; it has exactly 10 factors of 1.25.
34. No;  $(1.25)^{10}$  is about 9.3 and  $(1.25) \times 10$  is 12.5.
35. No;  $(1.25)^{10}$  is about 9.3 and  $(1.25) + 10 = 11.25$ .
36. Yes;  $(1.25^5)^2 = 1.25^5 \times 1.25^5$ , which has exactly 10 factors of 1.25.
37. No;  $1.25^5 \times 1.25^2$  has exactly seven factors of 1.25, so it is equal to  $(1.25)^7$ , not  $(1.25)^{10}$ .

38. Yes; it has exactly 7 factors of 1.5.
39. Yes; it has exactly 7 factors of 1.5.
40. No;  $(1.5)^7$  is about 17 and  $1.5 \times 7 = 10.5$ .
41. No;  $(1.5)^7$  is about 17 and  $1.5 + 7 = 8.5$ .
42. The graphs of  $y = 4^x$  and  $y = 10(4^x)$  have the same growth factor of 4, so they are both exponential growth patterns. The graphs  $y = 0.25^x$  and  $y = 10(0.25)^x$  are exponential decay patterns and have the same decay factor of 0.25. The graphs of  $y = 4^x$  and  $y = 0.25^x$  have a y-intercept of (0, 1). The graphs of  $y = 10(4^x)$  and  $y = 10(0.25)^x$  have y-intercepts (0, 10).
43. a. Graph B  
 b. Graph C

## Connections

44. False; because  $1.56892 \times 10^5 = 156,892$  is greater than  $2.3456 \times 10^4 = 23,456$ , the difference is greater than zero.
45. False; because  $3.96395 \times 10^5 = 396,395$  is less than  $2.888211 \times 10^7 = 28,882,110$ , the quotient is less than 1.
46. a. Volume: 8 units<sup>3</sup>; surface area: 24 units<sup>2</sup>.  
 The side lengths increase to 2 units. The new volume is  $2^3 = 2 \cdot 2 \cdot 2 = 8$  units<sup>3</sup>. Because there are six square faces, each with area  $2^2 = 4$ , the total surface area is  $6 \cdot 2^2 = 24$  units<sup>2</sup>.  
 b. Volume: 27 units<sup>3</sup>; surface area: 54 units<sup>2</sup>.  
 The side lengths increase to 3 units. The new volume is  $3^3 = 3 \cdot 3 \cdot 3 = 27$  units<sup>3</sup>, and the new surface area is  $6 \cdot 3^2 = 54$  units<sup>2</sup>.  
 c. Volume: 1,000,000 units<sup>3</sup>; surface area: 54 units<sup>2</sup>. The side lengths increase to 100 units each. The new volume is  $100^3 = 100 \cdot 100 \cdot 100 = 1,000,000$  units<sup>3</sup>, and the new surface area is  $6 \cdot 100^2 = 60,000$  units<sup>2</sup>.
47. a.  $8\pi$  units<sup>3</sup>. The resulting cylinder has a radius of 2 units and a height of 2 units, so the volume is  $\pi(2)^2 \times 2 = 8\pi$  units<sup>3</sup>.  
 b.  $27\pi$  units<sup>3</sup>. The resulting cylinder has a radius of 3 units and a height of 3 units, so the volume is  $\pi(3)^2 \times 3 = 27\pi$  units<sup>3</sup>.

- c.  $1,000,000\pi$  units<sup>3</sup>. The resulting cylinder has a radius of 100 units and a height of 100 units, so the volume is  $\pi(100)^2 \times 100 = 1,000,000\pi$  units<sup>3</sup>.
48. a. The following are prime:  $2^2 - 1 = 3$ ;  $2^3 - 1 = 7$ ;  $2^5 - 1 = 31$
- b. Other primes that fit this pattern include  $2^7 - 1 = 127$  and  $2^{13} - 1 = 8,191$ .
49. a. The sum of the proper factors of  $2^2$  is 3.
- b. The sum of the proper factors for  $2^3$ , or 8, is  $1 + 2 + 4 = 7$ .
- c. The sum of the proper factors for  $2^4$ , or 16, is  $1 + 2 + 4 + 8 = 15$ .
- d. The sum of the proper factors for  $2^5$ , or 32, is  $1 + 2 + 4 + 8 + 16 = 31$ .
- e. The sum of the proper factors of a power of 2 is always 1 less than the number.
50. Yes; the two towns will have the same populations if they continue to change at the same rates. Even though Grandville has a greater starting population, its population is decreasing, and in TINYTOWN, the population is increasing. So, eventually, the graphs will cross. However, it will take over 28 years for this to happen.
51. a. Possible answers:  $\frac{3(10)^5}{10^7}$ ,  $3 \times 10^{-2}$ , 0.03,  $\frac{3}{100}$ .
- b. Possible answers:  $\frac{5(10)^5}{2.5(10)^7}$ ,  $2 \times 10^{-2}$ , 0.02,  $\frac{2}{100}$ ,  $\frac{1}{50}$

## Extensions

52. a. Row 1:  $\frac{1}{2}$ , row 2:  $\frac{3}{4}$ , row 3:  $\frac{7}{8}$ , row 4:  $\frac{15}{16}$
- b.  $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + (\frac{1}{2})^4 + (\frac{1}{2})^5 = \frac{31}{32}$
- c.  $\frac{1,023}{1,024}$ ,  $\frac{1,048,575}{1,048,576}$
- d. The sum of each row is a fraction with a denominator equal to 2 raised to the power of that row number, and a numerator that is 1 less than the denominator. In the  $n$ th row, the sum will be  $\frac{2^n - 1}{2^n}$ .
- e. Row 4
- f. 1
- g. The pattern is similar to adding the areas of one of the ballots produced by each cut. It may appear that this total area will eventually equal the area of the original

sheet, but the pattern demonstrates that the total of the areas of the ballots will never actually equal the area of the whole piece.

53. a. Row 1:  $\frac{1}{3}$ , row 2:  $\frac{4}{9}$ , row 3:  $\frac{13}{27}$ , row 4:  $\frac{40}{81}$
- b.  $\frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + (\frac{1}{3})^4 + (\frac{1}{3})^5 = \frac{121}{243}$
- c. Row 10:  $\frac{29,524}{59,049}$ ; row 20:  $\frac{1,743,392,200}{3,486,784,401}$
- d. The sum of each row is a fraction with a denominator equal to 3 raised to the power of that row number, and a numerator that is half of the number that is 1 less than the denominator. Each row is a sum of powers of  $\frac{1}{3}$ . In the  $n$ th row, the sum will be  $\frac{3^n - 1}{3^n}$ .
- e. The sum seems to be approaching 0.5.
54. a.  $2^{-1} = \frac{1}{2}$ ,  $2^{-2} = \frac{1}{4}$ ,  $2^{-3} = \frac{1}{8}$
- b.  $(\frac{1}{2})^1 = \frac{1}{2}$ ,  $(\frac{1}{2})^2 = \frac{1}{4}$ ,  $(\frac{1}{2})^3 = \frac{1}{8}$
- c. Students should notice that  $2^x$  and  $(\frac{1}{2})^x$  have the same value when numbers of the opposite sign with the same absolute value are substituted. In other words,  $2^{-x} = (\frac{1}{2})^x$ .
- d.  $3^{-1} = (\frac{1}{3})^1$ ,  $4^{-2} = (\frac{1}{4})^2$ , and  $5^{-3} = (\frac{1}{5})^3$
55. a.

Standard Form	Exponential Form
10,000	$10^4$
1,000	$10^3$
100	$10^2$
10	$10^1$
1	$10^0$
$\frac{1}{10} = 0.1$	$10^{-1}$
$\frac{1}{100} = 0.01$	$10^{-2}$
$\frac{1}{1,000} = 0.001$	$10^{-3}$
$\frac{1}{10,000} = 0.0001$	$10^{-4}$
$\frac{1}{100,000} = 0.00001$	$10^{-5}$
$\frac{1}{1,000,000} = 0.000001$	$10^{-6}$

- b. 0.3, 0.015, 0.0015

56. a. 0.12, 0.012, 0.0012, 0.000000012
- b. Because  $1.2 \times 10^{-n} = 1.2 \times \frac{1}{10^n}$ , the standard form is 1.2 divided by the  $n$ th power of 10. When dividing by a power of 10, the decimal point in the number moves to the left. Because 1.2 is divided by the  $n$ th power of 10, the decimal place is moved to the left  $n$  places; thus,  $1.2 \times \frac{1}{10^n} = \underbrace{0.0000\cdots 0000012}_{n-1 \text{ zeros}}$ .

- c.  $2.0 \times 10^6$ ,  
 $2.8 \times 10^7$ ,  
 $1.99 \times 10^{10}$ ,  
 $1.2489 \times 10^{-1}$ ,  
 $5.8421998 \times 10^{-3}$ ,  
 $1.0201 \times 10^{-3}$

57. a. Possible answer:  $\frac{1.5 \times 10^{-4}}{10^8}$
- b. Possible answer:  $1.5 \times 10^{-4} \times 10^{-8}$

58. a. 1,740,000 meters
- b.  $1.0795 \times 10^1$
- c. The scale that would make the image fit exactly is  $6.204 \times 10^{-6}$ . Any scale factor smaller than this will make the image small enough to fit.

59. a.  $2^8 = 256$ , or  $2.56 \times 10^2$ .
- b.  $(\frac{1}{2})^8 = \frac{1}{256} = 2^{-8}$ , or  
 $0.00390625 = 3.90625 \times 10^{-3}$ .
- c.  $20^8 = 25,600,000,000$ , or  $2.56 \times 10^{10}$
- d.  $(\frac{1}{20})^8 = \frac{1}{25,600,000,000} = 20^{-8}$  or  
 $0.000000000390625 = 3.90625 \times 10^{-11}$

60. a.  $3\frac{1}{3}$ ,  $1\frac{1}{9}$ ,  $\frac{10}{27}$ ,  $\frac{10}{81}$ ,  $\frac{10}{243}$ ,  $\frac{10}{729}$ ,  $\frac{10}{2,187}$
- b. This means  $5.645029269 \times 10^{-5}$ . In standard notation, this is 0.00005645029269.
- c.  $4.57 \times 10^{-3}$ ,  $1.52 \times 10^{-3}$ ,  $5.08 \times 10^{-4}$ ,  
 $1.69 \times 10^{-4}$

61. a.

Number of Cuts	Area (sq. ft)
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
5	$\frac{1}{32}$
6	$\frac{1}{64}$
7	$\frac{1}{128}$
8	$\frac{1}{256}$

- b.  $A = (\frac{1}{2})^n$
- c. About  $9.54 \times 10^{-7}$  ft<sup>2</sup>. This doesn't make sense because a piece of paper could not be cut this small.

62. About \$86.08 per acre. The growth factor is 1.04. The cost has been inflating for 203 years ( $2006 - 1803 = 203$ ). Find the initial price per sq. mi:  
 $\$15,000,000 \div 828,000 \text{ sq. mi} \approx \$18.13$  per sq. mile. To get the initial price per acre, divide this value by 640:  $\$18.13$  per mile  $\div 640$  acres per mile  $\approx \$0.03$  per acre. Thus, the value of 1 acre of land in 2006 is  $(\$0.03)(1.04)^{20} \approx \$86.08$ .

63. a.  $\frac{1}{2}(2)^n = 2^{-1} \cdot 2^n = 2^{n-1}$
- b.  $4^n - 1 = 4^n \cdot 4^{-1} = 4^{-1} \cdot 4^n = \frac{1}{4}(4)^n$
- c.  $25(5^n - 2) = 5^2 \cdot 5^n - 2 = 5^{n-2+2} = 5^n$

## Possible Answers to Mathematical Reflections

1. To multiply powers with the same base, keep the same base and add the exponents:

$$a^m \times a^n = a^{m+n}$$

To multiply powers with the same exponent, multiply the bases and keep the exponent:

$$a^m \times b^m = (ab)^m$$

To raise a power to a power, keep the base and multiply the exponents:

$$(a^m)^n = a^{mn}$$

To divide powers of the same base, keep the base and use the numerator exponent minus the denominator exponent as the exponent:

$$\frac{a^m}{a^n} = a^{m-n}, \text{ for } a \neq 0$$

If the exponent in the denominator is greater than the exponent in the numerator, this division rule results in a negative exponent. There is a different form of the rule that always gives a positive exponent: If the exponent of the numerator is greater than the exponent of the denominator, then use the rule above. If the exponent of the denominator is greater than the exponent of the numerator, then the result is a fraction with a numerator of 1 and a base equal to the base raised to the denominator exponent minus the numerator exponent:

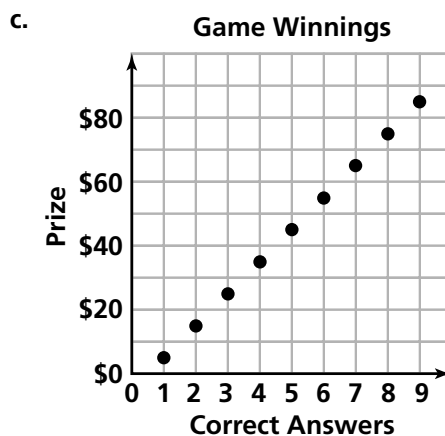
$$\begin{aligned} \frac{a^m}{a^n} &= a^{m-n} \text{ if } m \geq n \\ &= \frac{1}{a^{n-m}} \text{ if } m < n \end{aligned}$$

(Note: Students should know why these rules work.)

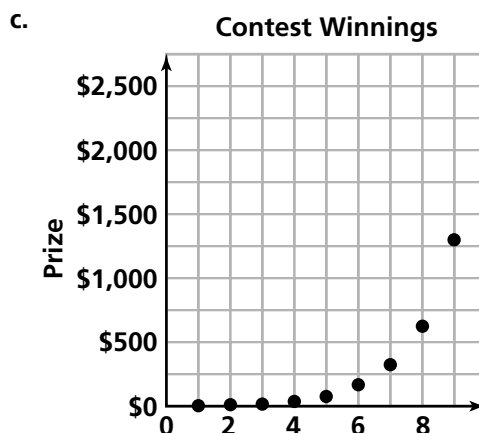
2. a. When  $b$  is greater than 1, the graph curves upward slowly at first and then very rapidly.  
b. When  $b$  is equal to 1, the graph is a horizontal line.  
c. When  $b$  is between 0 and 1, the graph curves downward rapidly at first, then more slowly, until it is almost horizontal and very close to the  $x$ -axis.
3. a. When  $a$  is greater than 1, the  $y$ -intercept is greater than 1.  
b. When  $a$  is equal to 1, the intercept is 1.  
c. When  $a$  is between 0 and 1, the  $y$ -intercept is less than 1.

## Answers to Looking Back and Looking Ahead

1. a.  $p = 10(n - 1) + 5$  or  $p = 10n - 5$ , for  $n \geq 1$ .  
b. 6 correct answers will give a prize of \$55.  
8 correct answers will give a prize of \$75.  
11 correct answers will give a prize of \$105.  
It is not possible to win exactly \$50 or exactly \$100.



2. a.  $p = 5(2^{n-1})$ , for  $n > 1$   
b. Four correct answers will give a prize of only \$40. Five correct answers will give a prize of \$80. Six correct answers will give a prize of \$160. It is not possible to win exactly \$50, \$75, or \$100.



- d. Possible answer: Lucy's proposal gives a linear relationship while Pedro's proposal is exponential. Contestants would probably prefer Pedro's proposal because they would get more money after three correct answers.
3. a.  $2^n$   
b.  $3,600(0.5)^n$
4. False;  $3^5 \times 6^5 = 18^5$ , which is not equal to  $9^5$ .
5. True;  $8^5 = 2^{15}$  and  $4^6 = 2^{12}$ , so  $8^5 \times 4^6 = 2^{15} \times 2^{12} = 2^{27}$ .
6. True;  $2^0 = 1$